HYDRODYNAMICS OF RELATIVISTIC SYSTEMS WITH BROKEN CONTINUOUS SYMMETRIES

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We show how hydrodynamics of relativistic system with broken continuous symmetry can be constructed using the Poisson bracket technique. We illustrate the method on the example of relativistic superfluids.

The study of hydrodynamics of relativistic systems is important for the physics of heavy-ion collisions. Indeed, hydrodynamic models are the conceptually simplest models of heavy-ion reactions.[1,2]

By definition, hydrodynamics is the effective theory describing the real-time dynamics of a given system at length scales larger than the mean free path and time scales larger than the mean free time. Implicitly in this definition, one assumes a nonzero temperature for the mean free path to be finite. At large length and time scales, only a small number of degrees of freedom survive to become hydrodynamic modes. If the system is far away from all second order phase transitions, the latter modes are divided into two categories: the conserved densities (i.e. densities of conserved charged), and the phases of order parameters. In systems with no broken continuous symmetry, all hydrodynamics modes belong to the first type, and the hydrodynamic equations are the conservation laws. But if there are broken continuous symmetries in the theory, then they give rise to new hydrodynamic modes. Our task is to find out the hydrodynamic equations that describe the coupled dynamics of Goldstone and fluid dynamical modes. This can be done using the Poisson bracket method described below.

For definiteness we will consider the simplest theory with a broken symmetry, the complex ϕ^4 field theory:

$$L = (\partial_\mu \phi^*)(\partial_\mu \phi) - \lambda(|\phi|^2 - v^2)^2$$

This theory is the relativistic generalization of superfluids. The set of hydrodynamic modes in this theory include five conserved densities: the energy density T^{00}, three components of the momentum density T^{0i}, and the density of the U(1) charge $n = -\frac{i}{2}(\phi^* \partial_0 \phi - \partial_0 \phi^* \phi)$, and one non-conserved phase φ, $\langle\phi\rangle = |\langle\phi\rangle| e^{i\varphi}$. If one ignore dissipative processes, then instead of T^{00} one can use the entropy density s.

At the operational level, the Poisson bracket method works as follows. We first write down the classical Poisson bracket between the hydrodynamic variables

$$
\begin{aligned}
[T^{0i}(\boldsymbol{x}),\, T^{0k}(\boldsymbol{y})] &= \left[T^{0k}(\boldsymbol{x})\frac{\partial}{\partial x^i} - T^{0i}(\boldsymbol{y})\frac{\partial}{\partial y^k}\right]\delta(\boldsymbol{x}-\boldsymbol{y}) \qquad (1)\\
[T^{0i}(\boldsymbol{x}),\, n(\boldsymbol{y})] &= n(\boldsymbol{x})\partial_i\delta(\boldsymbol{x}-\boldsymbol{y}) \qquad (2)\\
[T^{0i}(\boldsymbol{x}), s(\boldsymbol{y})] &= s(\boldsymbol{x})\partial_i\delta(\boldsymbol{x}-\boldsymbol{y}) \qquad (3)\\
[T^{0i}(\boldsymbol{x}), \varphi(\boldsymbol{y})] &= \partial_i\varphi\delta(\boldsymbol{x}-y) \qquad (4)\\
[n(\boldsymbol{x}), \varphi(\boldsymbol{y})] &= -\delta(\boldsymbol{x}-\boldsymbol{y}) \qquad (5)
\end{aligned}
$$

Eqs. (1,2) can be derived by direct calculation of quantum commutators, using the canonical commutation relation. Eqs. (3,4,5) are postulated from physical considerations. Eqs. (3,4) is consistent with the fact that $\int d\boldsymbol{x}\, T^{0i}(\boldsymbol{x})$ is the total momentum and generates a coordinate translation. The difference in the form of Eqs. (3) and (4) is due to the different dimensionalities of s and ϕ, which transform differently under dilatation $\int d\boldsymbol{x}\, x^i T^{0i}(\boldsymbol{x})$. Finally, Eq. (5) tells us that the charge and the phase of the condensate are conjugate variables.

The next step is to write down the most general Hamiltonian, which is also the total energy,

$$H = \int d\boldsymbol{x}\, T^{00}(s, n, T^{0i}, \partial_i\varphi) \qquad (6)$$

Note that T^{00} may depends only on the derivatives of φ but not on φ itself due to the invariance under the U(1) rotation. The form of the function T^{00} is undefined at this moment and can be completely found only by calculations at the microscopic level. However some constraint on the form of T^{00} follows from relativistic invariance, see below. Knowing the Hamiltonian (6) and the Poisson brackets (1-5), the classical dynamics of the hydrodynamic modes are fixed completely. The equation of motion for any variable A has the form

$$\dot{A} = [H, A] \qquad (7)$$

In particular, one can find the equation of motion for the energy density T^{00}, $\dot{T}^{00} = [H, T^{00}]$. From relativistic invariance one expects that $\dot{T}^{00} = -\partial_i T^{0i}$: the energy flux is equal to the momentum density. This imposes a severe constraint on the possible forms of the function $T^{00}(s, n, T^{0i}, \partial_i\varphi)$, and makes possible a relativistically covariant formulation of hydrodynamics. Omitting computational details, the final formulation is as follows.

First, one should find, from microscopic physics, the equation of state, which define the pressure as a function of three variables,

$$p = p(T, \mu, \frac{1}{2}(\partial_\mu\varphi)^2) \qquad (8)$$

where the first two variables are nothing but the temperature and the chemical potential of the conserved charge, and the third variable is specific for the superfluid phase and defines the degree of variation of the U(1) condensate phase over space-time. Knowing the equation of state (8), one defines the thermodynamic variables conjugate to T, μ and $\frac{1}{2}(\partial_\mu\varphi)$:

$$dp = sdT + nd\mu + V^2 d(\frac{1}{2}(\partial_\mu\varphi)^2) \tag{9}$$

One also defines ρ as the Legendre transform of $-p$ with respect to T and μ, $\rho = sT + n\mu - p$. The physical meaning of s, n and V is clear from the hydrodynamic equations which have the form

$$\partial_\mu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = (\rho+p)u^\mu u^\nu - pg^{\mu\nu} + V^2\partial^\mu\varphi\partial^\nu\varphi \tag{10}$$

$$\partial_\mu(nu^\mu - V^2\partial^\mu\varphi) = 0 \tag{11}$$

$$\partial_\mu(su^\mu) = 0 \tag{12}$$

$$u^\mu\partial_\mu\varphi + \mu = 0 \tag{13}$$

Eq. (10) is energy-momentum conservation, where the energy-momentum tensor $T^{\mu\nu}$ consists of two parts: a "fluid" part that is due to the particles outside the condensate, and a "field" part which is due to the coherent motion of the condensate. Similarly, Eq. (11) is conservation of the U(1) charge, where the conserved current is also a sum of a normal current and a superfluid current. The velocity u^μ is clearly the velocity of the normal component of the fluid. Eq. (12) is the conservation law for entropy: only the normal component contributes to the entropy flow, in accordance with physical expectations. Finally Eq. (13) is not a conservation law but a Josephson-type equation that relate the time derivative of the condensate phase in the frame where no normal flow occurs with the chemical potential.

It can be shown that Eqs. (10-13) are equivalent (although only after a nontrivial mapping) to the set of equations previously suggested by Carter, Khalatnikov and Lebedev for relativistic superfluids.[3,4] The advantage of the form (10-13) is that the physical meaning and the field-theoretical origin of all quantities and equations are made quite clear.

One can also generalize this technique to the case of nuclear matter. Near the chiral limit, nuclear matter possesses a broken continuous symmetry, which is $SU(N_f) \times SU(N_f) \to SU(N_f)$. The hydrodynamics in this case described the evolution of $5 + 2(N_f^2 - 1)$ conserved charges (energy and momentum, entropy, and left and right-handed isospin charges) and $N^2 - 1$ condensate phases (the Goldstone bosons). For details, see Ref. [5].

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